[13.31] Let ***V*** be a vector space and ***T*** be a linear transformation on ***V*** with distinct eigenvalues **1, …, **m, where *m* ≤ n. We furthermore assume

1. For each *j* of multiplicity  ≥ 2 (if any), there are  independent eigenvectors.

Prove there is a basis for ***V*** composed of eigenvectors.

**Solution**. Let  be the multiplicity of eigenvalue *j*. Since there are *n* eigenvalues, we have that  Let B *j* =  be the set of  independent eigenvectors corresponding to *j*. We wish to prove 

comprises a basis for ***V***. Since Bcontains *n* vectors, it suffices to show that these vectors are linearly independent. So, assume



We will be done if we can show that  so suppose some  We show this leads to a contradiction, which will complete the proof.

Since the double sum (\*) has a finite number of terms, there is some collection  of non-zero coefficients satisfying (\*) having as few terms as possible. That is,  and a set 

such that

 has the minimum number of terms.

If *p* = 1, all the eigenvectors arise from a single eigenvalue and are thus independent by condition (a). Hence  contradicting that they are all non-zero.

So, we assume *p* > 1. Since we can apply *T* to equation (1) to get



Multiplying equation (1) by *p* gives



Subtracting (3) from (2) gives





In equation (4),  and the eigenvalues are distinct. Thus we have produced a shorter relation than (1), yielding the afore-mentioned contradiction, completing the proof. ✔

**Corollary**. In the basis B of eigenvectors, *T* is represented by a diagonal matrix with the Eigenvalues on the diagonal.

Proof. Re-label  and re-label the corresponding eigenvalues  (For clarification, in this notation if there are multiple eigenvalues, then we will have  for come cases where *i* ≠ *j*.) In the basis  *T* takes a diagonal form of Eigenvalues because  or ∀ k



 ✔